Quantum effects in the evolution of vortices in the electromagnetic field

Tomasz Radożycki*
Department of Mathematical Methods in Physics, Faculty of Physics, Warsaw University, Hoża 74, 00-682 Warsaw, Poland
(Received 20 January 2004; published 18 June 2004)

We analyze the influence of electron-positron pairs creation on the motion of vortex lines in electromagnetic field. In our approach the electric and magnetic fields satisfy nonlinear equations derived from the Euler-Heisenberg effective Lagrangian. We show that these nonlinearities may change the evolution of vortices.

DOI: 10.1103/PhysRevE.69.066616 PACS number(s): 03.50.De, 67.40.Vs, 11.10.Lm, 11.27.+d

I. INTRODUCTION

The phenomena of creation and evolution of vortices have always attracted attention both in the past and in the present. In contemporary physics they gained particular interest since having been experimentally observed in Bose-Einstein condensates [1–4]. Vortices in superfluids, due to the absence of viscosity, exhibit certain unconventional features like the persistence of the whirl or its singular nature. The Bose-Einstein condensate may be described by the nonlinear Schrödinger equation (the Gross-Pitaevskii equation [5–7]) satisfied by a certain macroscopic wave function. In that way one has been led to studying vortices in quantum mechanics (QM). There is a striking resemblance between the dynamics of fluids and QM via the hydrodynamic formulation of the latter [8]. QM can, even in the linear version, serve as a model theory for investigating the behavior of vortices in superfluids. Such studies, concerned with the dynamical as well as topological aspects of vortex evolution in various configurations, both in nonlinear [9–18] and in linear [19–23] cases, have recently been undertaken.

Together with the attention paid to nonrelativistic QM, the singular solutions in other field theories such as electromagnetism, for instance, have been investigated [24–28]. We will be concerned with this question also in the present paper. While considering vortices in fields corresponding to spinning particles one encounters the problem that the wave funcion has more than one component and the condition $\psi(r,t)=0$, leads to too many equations which cannot be simultaneously satisfied. One of the solutions of this problem for electromagnetic field was proposed in Ref. [27], where vortex lines were defined by the null values of two relativistic invariants: $\mathcal{S}$ and $\mathcal{P}$. These two equations mean two surfaces in the three-dimensional space. Their intersection in general may be a curve — a vortex line. We will base our work on this approach.

In the present work we would like to investigate how quantum effects can influence the motion of the nodal lines of the electromagnetic wave function [29]. The term "quantum" is used here in the field theoretical sense: Maxwell electrodynamics, as well as Schrödinger wave mechanics, are classical from that point of view. We will concentrate on the effect of electron-positron pairs creation in electromagnetic fields. In order to construct the position and time dependent wave function, we still need the classical equations for electric and magnetic fields. These equations are no longer linear since pairs creation leads to the photon-photon interaction and Maxwell equations are suplemented by additional terms which, in the lowest approximation, are cubic in fields and quadratic in the fine-structure constant (in general they might be also nonlocal). Although the correction is small it can certainly influence the motion of the vortex lines and particularly change their topology.

One should mention here that there exists also another type of quantum effects — which remain beyond the concern of the present work — connected not with the e^+e^- content of the vacuum, but with fluctuations of the electromagnetic field itself. These effects, due to the nonzero vacuum expectation value of billinears in fields, lead to the smoothing of the vortex core (i.e., the line on which both invariants are equal to zero). In this case the core is no more singular. It is defined not by the condition $F^2=0$, which is not satisfied, but rather $F^2\approx 0$, where $\approx$ means $|F|^2_{classical}<|F|^2_{vac.fluct.}$ [30], where F is the Riemann-Silberstein vector spoken more of in Secs. II and III.

As a starting point we choose the Euler-Heisenberg (EH) Lagrangian [31,32] describing, in the lowest order, the dynamics of classical electromagnetic fields with vacuum polarization effects taken into account. Field equations obtained from this effective theory in Sec. II exhibit solutions containing vortex lines, the evolution of which may be viewed and compared to that obtained from the classical Maxwell equations. In this work we analyze two such cases. Both are chosen from Ref. [27] to make the comparison of the results in our works very easy. Our results are presented in Sec. III.

The main practical problem in this investigation comes from the fact that quantum corrections are, in general, small and it is very hard to see them on a drawing. The choice of examples considered in our work from among those of Ref. [27] is dictated just by the criterion of quantum effects being visible. It is clear that they are noticeable not by analyzing or measuring the precise shape of a vortex line — the slight deviation of which from that obtained in classical theory surely does occur — but rather by observing "to be or not to be" effects or topological ones.

There are two limitations which cause some of the results of this work to be qualitative rather than quantitative. First, they are obtained within perturbative regime. This regime means that the electromagnetic field may not be too strong and its strength is limited by the condition of $\alpha|\boldsymbol{F}|^2/m^4$ being

*Electronic address: torado@fuw.edu.pl

small. Fields considered here are polynomial, so this requirement means that the evolution should not go beyond a certain limited space-time region. However, close to the vortex line, similarly to the situation in the Gross-Pitaevskii equation, the terms quadratic in $\mathcal{S}$ and $\mathcal{P}$ in the Hamiltonian (9) become small, even for large values of electromagnetic fields, and perturbative calculation is again well justified.

Secondly, we have to remember that the EH Lagrangian describes only slowly varying fields, for which the nonlocality may be neglected. Their relative change at a distance of the Compton wavelength of the electron should be small. In view of that the EH effective Lagrangian is treated in our work as a certain nonlinear model of the true theory of electromagnetic fields obtained from QED without real charges. Another interesting model in this context constitutes the Born-Infeld electrodynamics [33]. One should, however, have in mind that even small corrections, coming from weak fields, can change the topology of vortices.

II. FIELD EQUATIONS

The Euler-Heisenberg Lagrangian [31,32], which accounts for the vacuum polarization processes in the lowest approximation has the following form:

$$\mathcal{L}(\boldsymbol{r},t)=\mathcal{S}(\boldsymbol{r},t)+\frac{2\alpha^2}{45m^4}[4\mathcal{S}(\boldsymbol{r},t)^2+7\mathcal{P}(\boldsymbol{r},t)^2], \tag{1}$$

where $\mathcal{S}$ and $\mathcal{P}$ denote the two Poincaré invariants formed of electromagnetic fields,

$$\mathcal{S}=\frac{1}{2}(\boldsymbol{E}^2-\boldsymbol{B}^2),\quad \mathcal{P}=\boldsymbol{E}\cdot\boldsymbol{B}, \tag{2}$$

and α and m are fine-structure constant and electron mass, recpectively. The canonical variables are electric and magnetic inductions $\boldsymbol{D}$ and $\boldsymbol{B}$, where the former plays the role of canonical momentum and the latter of position [33,34]. In this picture the electric field strength $\boldsymbol{E}$ in the Lagrangian (1) corresponds to the velocity. The field equations that we will need for our purpose are the canonical Hamilton equations

$$\dot{\boldsymbol{D}}(\boldsymbol{r},t)=\boldsymbol{\nabla}\times\frac{\partial\mathcal{H}(\boldsymbol{r},t)}{\partial\boldsymbol{B}(\boldsymbol{r},t)}, \tag{3a}$$

$$\dot{\boldsymbol{B}}(\boldsymbol{r},t)=-\boldsymbol{\nabla}\times\frac{\partial\mathcal{H}(\boldsymbol{r},t)}{\partial\boldsymbol{D}(\boldsymbol{r},t)}, \tag{3b}$$

where $\mathcal{H}(\boldsymbol{r},t)$ denotes the Hamiltonian density. To find the explicit form of Eqs. (3) we have to perform the Legendre transform and pass from $\mathcal{L}$ to $\mathcal{H}$. The canonical momentum is, as always, defined as a derivative of the Lagrangian over velocity,

$$\boldsymbol{D}(\boldsymbol{r},t)=\frac{\partial\mathcal{L}(\boldsymbol{r},t)}{\partial\boldsymbol{E}(\boldsymbol{r},t)}=\frac{\partial\mathcal{L}(\boldsymbol{r},t)}{\partial\mathcal{S}(\boldsymbol{r},t)}\boldsymbol{E}(\boldsymbol{r},t)+\frac{\partial\mathcal{L}(\boldsymbol{r},t)}{\partial\mathcal{P}(\boldsymbol{r},t)}\boldsymbol{B}(\boldsymbol{r},t), \tag{4}$$

which gives

$$\boldsymbol{D}(\boldsymbol{r},t)=\left[1+\frac{16\alpha^2}{45m^4}\mathcal{S}(\boldsymbol{r},t)\right]\boldsymbol{E}(\boldsymbol{r},t)+\frac{28\alpha^2}{45m^4}\mathcal{P}(\boldsymbol{r},t)\boldsymbol{B}(\boldsymbol{r},t). \tag{5}$$

This kind of equation usually bears the name of a constitutive equation. It reflects the nontrivial structure of the medium. In the present case this medium is the quantum field theory vacuum with its polarizability via electron-positron pairs creation and anihilation. We now need to invert this equation and express velocity $\boldsymbol{E}$ in terms of canonical variables $\boldsymbol{D}$ and $\boldsymbol{B}$. Since our initial Lagrangian (1) is given only in one loop approximation (α^2) then our further calculations may be led up to this order too. We can therefore postulate $\boldsymbol{E}(r,t)$ in the form

$$\boldsymbol{E}(\boldsymbol{r},t)=[1+\alpha^2\mathcal{K}(\boldsymbol{r},t)]\boldsymbol{D}(\boldsymbol{r},t)+\alpha^2\mathcal{M}(\boldsymbol{r},t)\boldsymbol{B}(\boldsymbol{r},t), \tag{6}$$

where quantities $\mathcal{K}(\boldsymbol{r},t)$ and $\mathcal{M}(\boldsymbol{r},t)$ are to be determined. Substituting Eq. (6) into Eq. (5), neglecting terms of the order higher than α^2, and comparing coefficients multiplying vectors $\boldsymbol{D}$ and $\boldsymbol{B}$ we find

$$\mathcal{K}(\boldsymbol{r},t)=-\frac{16}{45m^4}[\boldsymbol{D}(\boldsymbol{r},t)^2-\boldsymbol{B}(\boldsymbol{r},t)^2], \tag{7a}$$

$$\mathcal{M}(\boldsymbol{r},t)=-\frac{28}{45m^4}\boldsymbol{D}(\boldsymbol{r},t)\cdot\boldsymbol{B}(\boldsymbol{r},t). \tag{7b}$$

The Hamiltonian density may be now found as

$$\mathcal{H}(\boldsymbol{r},t)=\boldsymbol{E}(\boldsymbol{r},t)\cdot\boldsymbol{D}(\boldsymbol{r},t)-\mathcal{L}(\boldsymbol{r},t), \tag{8}$$

where $\boldsymbol{E}$ in the whole above expression should be eliminated in favor of $\boldsymbol{D}$ and $\boldsymbol{B}$, according to the relations (6) and (7a). The explicit form of $\mathcal{H}$ is then

$$\mathcal{H}(\boldsymbol{r},t)=\frac{1}{2}[\boldsymbol{D}(\boldsymbol{r},t)^2+\boldsymbol{B}(\boldsymbol{r},t)^2]-\frac{2\alpha^2}{45m^4}[\boldsymbol{D}(\boldsymbol{r},t)^2-\boldsymbol{B}(\boldsymbol{r},t)^2]^2-\frac{14\alpha^2}{45m^4}[\boldsymbol{D}(\boldsymbol{r},t)\cdot\boldsymbol{B}(\boldsymbol{r},t)]^2. \tag{9}$$

Now we are in a position to write down equations (3) in an explicit form,

$$\dot{\boldsymbol{D}}(\boldsymbol{r},t)=\boldsymbol{\nabla}\times\left\{\boldsymbol{B}(\boldsymbol{r},t)\left[1+\frac{8\alpha^2}{45m^4}[\boldsymbol{D}(\boldsymbol{r},t)^2-\boldsymbol{B}(\boldsymbol{r},t)^2]\right]-\frac{28\alpha^2}{45m^4}\boldsymbol{D}(\boldsymbol{r},t)[\boldsymbol{D}(\boldsymbol{r},t)\cdot\boldsymbol{B}(\boldsymbol{r},t)]\right\}, \tag{10a}$$

$$\dot{\boldsymbol{B}}(\boldsymbol{r},t)=-\boldsymbol{\nabla}\times\left\{\boldsymbol{D}(\boldsymbol{r},t)\left[1-\frac{8\alpha^2}{45m^4}[\boldsymbol{D}(\boldsymbol{r},t)^2-\boldsymbol{B}(\boldsymbol{r},t)^2]\right]-\frac{28\alpha^2}{45m^4}\boldsymbol{B}(\boldsymbol{r},t)[\boldsymbol{D}(\boldsymbol{r},t)\cdot\boldsymbol{B}(\boldsymbol{r},t)]\right\}. \tag{10b}$$

Introducing two complex vectors $\boldsymbol{F}_{\pm}(\boldsymbol{r},t)$ according to the relation

$$\boldsymbol{F}_{\pm}(\boldsymbol{r},t)=\frac{1}{\sqrt{2}}[\boldsymbol{D}(\boldsymbol{r},t)\pm i\boldsymbol{B}(\boldsymbol{r},t)], \tag{11}$$

we can rewrite Eqs. (10) in the form

$$\dot{\boldsymbol{F}}_{+}(\boldsymbol{r},t)=-i\boldsymbol{\nabla}\times\boldsymbol{F}_{+}(\boldsymbol{r},t)+\frac{2i\alpha^2}{45m^4}\boldsymbol{\nabla}\times\{\boldsymbol{F}_{-}(\boldsymbol{r},t)[11\boldsymbol{F}_{+}(\boldsymbol{r},t)^2-3\boldsymbol{F}_{-}(\boldsymbol{r},t)^2]\}, \tag{12a}$$

$$\dot{\boldsymbol{F}}_{-}(\boldsymbol{r},t)=i\boldsymbol{\nabla}\times\boldsymbol{F}_{-}(\boldsymbol{r},t)+\frac{2i\alpha^2}{45m^4}\boldsymbol{\nabla}\times\{\boldsymbol{F}_{+}(\boldsymbol{r},t)[11\boldsymbol{F}_{-}(\boldsymbol{r},t)^2-3\boldsymbol{F}_{+}(\boldsymbol{r},t)^2]\}. \tag{12b}$$

In the classical case the right hand sides of Eqs. (12a) and (12b) reduce to the first terms only and the two equations for $\boldsymbol{F}_{\pm}$ decouple from each other. This is not the case in the presence of a nonlinear medium.

The evolution takes place in an empty space, without real charges, so $\boldsymbol{F}_{\pm}(\boldsymbol{r},t)$ have to satisfy the conditions

$$\boldsymbol{\nabla}\cdot\boldsymbol{F}_{\pm}(\boldsymbol{r},t)=0. \tag{13}$$

By applying gradient to both sides of Eqs. (12a) and (12b) it can easily be seen that $\boldsymbol{\nabla}\cdot\boldsymbol{F}_{\pm}(\boldsymbol{r},t)$ are constant in time and it is sufficient to impose the conditions (13) at time $t=0$.

III. EVOLUTION OF EXEMPLARY VORTICES

In the present section we would like to show how quantum effects connected with pairs creation influence the evolution of vortices in the electromagnetic field. From among the configurations of vortex lines considered in Ref. [27] we have chosen two, for which the comparison can most easily be done and the effects are clearly visible. They are the situations presented in Figs. 1 and 2 of Ref. [27]: the motion of the vortex ring and the creation and further evolution of initially linear vortex-antivortex configuration, i.e., two vortices of opposite whirl.

Vortex lines in quantum mechanics are usually defined by the behavior of the wave function of the system. In hydrodynamics vortices appear in the regions of space where $\boldsymbol{\nabla}\times\boldsymbol{v}(\boldsymbol{r},t)\neq\boldsymbol{0}$, where $\boldsymbol{v}(\boldsymbol{r},t)$ is the local fluid velocity. In QM, in its hydrodynamic formulation [2], the role of the fluid is played by the distrubution of probability. The velocity field, being proportional to the gradient of the phase of the wave function, can have nonvanishing curl only where this phase is singular. This in turn means the vanishing of the wave function, i.e., the simultaneous vanishing of its real and imaginary parts. In that way we are led to the conclusion that, in general, vortices have the character of the curves (evolving in time) costituting the intersection of two surfaces defined by the requirement $\psi(\boldsymbol{r},t)=0$.

As it was proposed in Ref. [27] one can introduce in electrodynamics, in place of ψ, a similar object, the vanishing of which may serve as the definition for the vortex lines. This object is $\boldsymbol{F}(\boldsymbol{r},t)^2$, where $\boldsymbol{F}(\boldsymbol{r},t)=(1/\sqrt{2})[\boldsymbol{D}(\boldsymbol{r},t)+i\boldsymbol{B}(\boldsymbol{r},t)]$. As argued [29], the quantity $\boldsymbol{F}$ is worthy of being called a "photon wave function." In the case considered in the present work, photons move in the polarizable medium, and what is more, the nonlinear one. Already in the linear (but inhomogeneous) medium one is forced to define the wave function as an extention of $\boldsymbol{F}$ through the introduction of upper and lower components [29] defined by Eq. (11),

$$\mathcal{F}(\boldsymbol{r},t)=\begin{pmatrix}\boldsymbol{F}_{+}(\boldsymbol{r},t)\\ \boldsymbol{F}_{-}(\boldsymbol{r},t)\end{pmatrix}. \tag{14}$$

This allows one to give the set of coupled equations the form of one, linear, Schrödinger-type equation for the wave function $\mathcal{F}(\boldsymbol{r},t)$. In the quantum case the linearity is inevitably lost, but the definition of vortex lines, by the requirements $\mathcal{S}(\boldsymbol{r},t)=0$ and $\mathcal{P}(\boldsymbol{r},t)=0$, seems to be universal (following Ref. [27] this kind of singular lines has recently been called "Riemann-Silberstein" vortices [35,36]). Therefore, in the full analogy with Ref. [27], we choose as a basic object the quantity

$$\boldsymbol{F}_{+}^2=\frac{1}{2}(\boldsymbol{D}^2-\boldsymbol{B}^2)+i\boldsymbol{D}\cdot\boldsymbol{B}. \tag{15}$$

The condition $\boldsymbol{F}_{+}(\boldsymbol{r},t)^2=\boldsymbol{0}$ is naturally equivalent to the choice $\boldsymbol{F}_{-}(\boldsymbol{r},t)^2=\boldsymbol{0}$.

A. Vortex ring

The first configuration considered in Ref. [27] is defined by

$$\boldsymbol{f}^{(a)}(\boldsymbol{r},t)=(y+it,z-a+i(a+t),x+it). \tag{16}$$

This "wave function" satisfies the Maxwell equations and describes the evolution of a single vortex in the form of a swinging ring with varying radius. In order to see in an easy way how quantum (nonlinear) terms in Eqs. (12a) and (12b) influence this evolution, we will choose the solution $\boldsymbol{F}_{+}(\boldsymbol{r},t)$ of Eqs. (12a) and (12b) which is identical to Eq. (16) at $t=0$. This solution (up to α^2) has the form

$$\boldsymbol{F}_{+}^{(a)}(\boldsymbol{r},t)=\boldsymbol{f}^{(a)}(\boldsymbol{r},t)+t^3\cdot\boldsymbol{\alpha}(\boldsymbol{r})+t^2\cdot\boldsymbol{\beta}(\boldsymbol{r})+t\cdot\boldsymbol{\gamma}(\boldsymbol{r}), \tag{17}$$

where vector funtions $\boldsymbol{\alpha}(\boldsymbol{r})$, $\boldsymbol{\beta}(\boldsymbol{r})$, and $\boldsymbol{\gamma}(\boldsymbol{r})$ are given by

$$\boldsymbol{\alpha}(\boldsymbol{r})=-\frac{128i\alpha^2}{135m^4}(1,1,1), \tag{18a}$$

$$\boldsymbol{\beta}(\boldsymbol{r})=\frac{8\alpha^2}{3m^4}\left(\frac{1}{3}(z-a)-\frac{2}{5}y-\frac{i}{3}a,\frac{2}{5}(a-z)+\frac{1}{3}x-\frac{i}{3}a,-\frac{2}{5}x+\frac{1}{3}y\right), \tag{18b}$$

$$\boldsymbol{\gamma}(\boldsymbol{r})=\frac{8\alpha^2}{3m^4}\left(\frac{2}{3}a(z-a)-\frac{i}{15}(11a^2-12az+6z^2),-\frac{2i}{5}x^2,-\frac{2i}{5}y^2\right). \tag{18c}$$

In Fig. 1 we show the evolution of the vortex line constituting the intersection of two surfaces $\mathrm{Re}\boldsymbol{F}_{+}^{(a)}(\boldsymbol{r},t)^2=0$ and

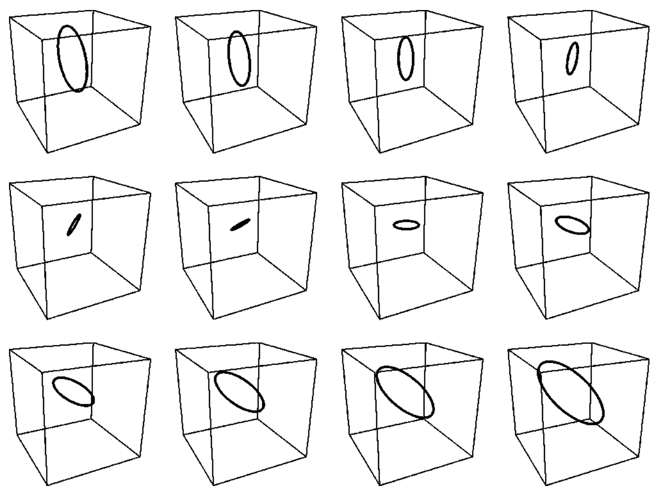

FIG. 1. The evolution of the vortex ring starting from $t=-1.8$ to $t=1.5$. The scale on the axes is such that the frame covers the region $-4<x,y,z<4$.

$\mathrm{Im}\,\boldsymbol{F}_+^{(a)}(\boldsymbol{r},t)^2=0$ with $\boldsymbol{F}_+^{(a)}(\boldsymbol{r},t)$ defined by Eqs. (17) and (18).

For simplicity both parameters a and m are set equal to unity on this, as well as on the following plots. The evolution extends in time from $t=-1.8$ to $t=1.5$ and exhibits identical character to that of Ref. [27]: the swinging vortex ring, preserving its circular character, decreases to a certain minimal value of radius, and then starts to increase. Quantum effects do not manifest themselves in this domain of space and time. In the classical case, hovever, the expansion of a ring will last forever, which can easily be seen from the two equations given in Ref. [27],

$$x^2+y^2+(z-a)^2-a^2-2at-3t^2=0, \tag{19a}$$

$$2az+2t(x+y+z-a)-2a^2=0. \tag{19b}$$

The former represents the sphere of a fixed center in the point $(0,0,a)$ and of constantly increasing radius (for positive t). The latter, rewritten in the form $x+y+(1+a/t)z=a(1+a/t)$, tends to the motionless plane $x+y+z=a$ passing through the center of the sphere. Their intersection will surely be the expanding ring. As we see in Fig. 2, this ceases to be true in the quantum case.

Due to the nonlinearity introduced by quantum effects two new phenomena appear. First, the vortex ring starts to deviate, for intermediate times, from its regular, circular character. Second, it is no longer constantly expanding. On the contrary, after reaching certain maximal extention it starts to decrease down to its complete disappearance, if we draw also frames for larger times.

If we traced the vortex evolution even further in time (certainly far beyond the applicability of the perturbative methods) we would observe the complicated system of vortices approaching from "infinity."

B. Vortex antivortex

The second situation corresponds to case (b) of Ref. [27],

$$\boldsymbol{f}^{(b)}(\boldsymbol{r},t)=(y+t,a-i(z+a-t),x+it). \tag{20}$$

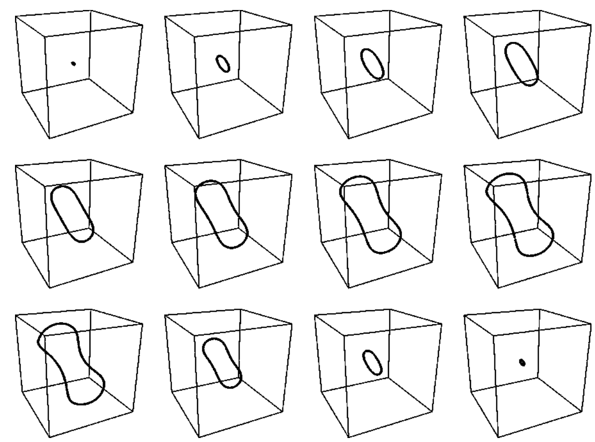

FIG. 2. The evolution of the vortex ring for larger times: from $t=1.5$ to $t=99.3$. The first frame is identical to the last one of Fig. 1, but now the scale of the axes is modified to $-100<x,y,z<100$.

The above function was shown to describe the configuration of two vortices which initially are antiparallel straight lines (we call them vortex and antivortex). They are born at $t=a$ and then they separate and deform. The solution of Eq. (12a) which is identical to $\boldsymbol{f}^{(b)}$ at $t=0$ has (again up to α^2) the form similar to Eq. (17),

$$\boldsymbol{F}_+^{(b)}(\boldsymbol{r},t)=\boldsymbol{f}^{(b)}(\boldsymbol{r},t)+t^3\cdot\boldsymbol{\alpha}(\boldsymbol{r})+t^2\cdot\boldsymbol{\beta}(\boldsymbol{r})+t\cdot\boldsymbol{\gamma}(\boldsymbol{r}), \tag{21}$$

but now with

$$\boldsymbol{\alpha}(\boldsymbol{r})=-\frac{8\alpha^2}{15m^4}\left(2,\frac{17i}{9},\frac{17i}{9}\right), \tag{22a}$$

$$\boldsymbol{\beta}(\boldsymbol{r})=\frac{8\alpha^2}{15m^4}\left(2(z-y+a)+\frac{5i}{3}a,\frac{5}{3}(x-a)+2i(z+a),-2x+2iy\right), \tag{22b}$$

$$\boldsymbol{\gamma}(\boldsymbol{r})=-\frac{8\alpha^2}{15m^4}\left(\frac{11}{3}a^2+4az+2z^2+\frac{10i}{3}a(z+a),2ix^2,2iy^2\right). \tag{22c}$$

The evolution of this vortex configuration is presented in Fig. 3. Again we have put $a=1$ and $m=1$.

We see in general the same motion as that found in Ref. [27] except one difference visible in the first frame. In Ref. [27] the two straight, antiparallel vortices spring up at $t=a$ (this time corresponds to the first frame) as exactly overlapping. No vortices exist for t between $-a$ and a. In the case of Fig. 3 the vortices in the first frame are slightly shifted and of different slope. This is a result of the influence of the nonlinear (quantum) terms in Eqs. (12a) and (12b) [38]. We recall that the vectors $\boldsymbol{F}^{(b)}(\boldsymbol{r},t)$ and $f^{(b)}(\boldsymbol{r},t)$ are synchronized for $t=0$ and not for $t=a$. This small shift and deformation are then consequences of the quantum correction to the evolution for $0<t<a$.

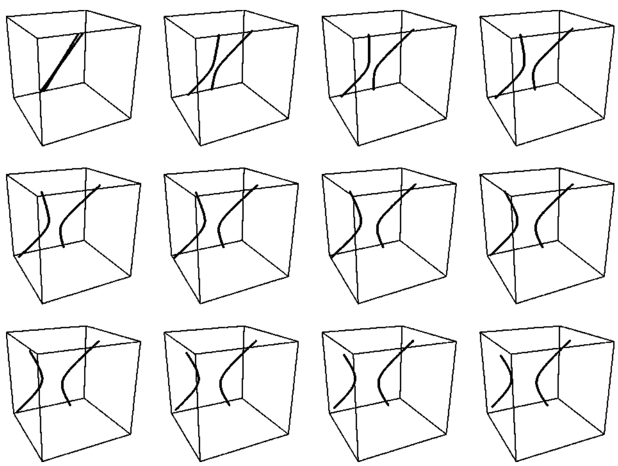

FIG. 3. The evolution of the system of two "antiparallel" vortices for time between $t=1$ and $t=1.55$. The units on the axes are such that each frame represents the cube $-4<x,y,z<4$.

In Ref. [27] the system of vortices is born at $t=a$, but in our case they do not overlap and consequently must have appeared earlier. It is therefore interesting to take a step back in time and see how these vortices arise in the quantum case. This is shown in Fig. 4.

Figure 4 brings to light the essential change: the two independent vortices in the classical case, or rather vortex and antivortex, become the two fractions of the same, tightly bent, vortex line, when quantum corrections are taken into account. Their sudden creation turns out now to be a motion during which this single vortex line simply enters into the observation region and is being deformed. One might expect this kind of effect — that could be called topological effects — together with the smoothing of the evolution, to be the most typical ones introduced by the nonlinearity of the quantum equations. To make the effect more visible we present it again in Fig. 5, now seen from another viewpoint.

We would like also to emphasize that the above phenomena take place for electromagnetic fields weak enough to remain in full agreement with the use of perturbation theory.

Yet another difference not visible, however, in Figs. 4 and 5, is the slight deviation of the system of vortex lines from

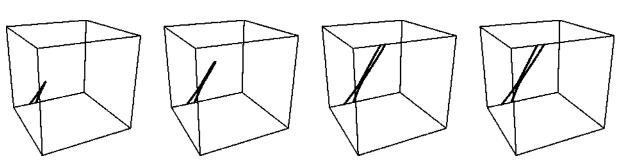

FIG. 4. The appearance of the system of vortices of Fig. 3. The frames correspond to times just before $t=a$. Now the scale on the y axis is changed to make the splitting of vortices easily visible: $-4<x,z<4$ and $-1.5<y<0.5$.

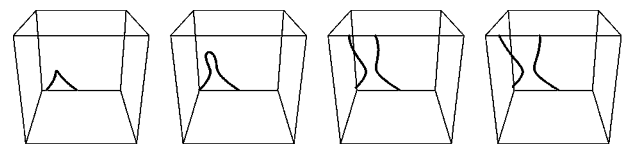

FIG. 5. The appearance of the system of vortices of Fig. 3 seen now from a viewpoint other than that of Fig. 4 and with the y axis rescaled even more.

their planar character. In the clasical case the vortex lines arose as the intersection of a plane with a certain surface, and therefore all vortices have to lie forever in one plane. This is no longer true in the quantum case.

IV. SUMMARY

In the present paper we analyzed the influence of the nonlinear, quantum terms in the Maxwell equations on the evolution of vortex lines. By making the comparison with the results obtained earlier in the classical case [27] we found that this evolution may be changed in a visible and essential way. In the first considered configuration of the constantly expanding vortex ring, our calculations show that quantum corrections may lead to the deformation and disappearance of this ring. In the second case of two linear and antiparallel vortices of the infinite size, which are suddenly created, we show what the process of this "creation" looks like, and that the two independent vortices (in the classical case) turn out to be just different fractions of the same vortex curve. This kind of topological change might be expected as a result of nonlinearity introduced by vacuum polarization.

The present analysis has certain limitations which come both from its perturbative character and from the "low frequency" approximation which allows one to derive the EH Lagrangian. It can, however, serve as a qualitative picture of what type of phenomena may be introduced by the quantum effects. One is still very far from constructing the nonperturbative solutions of quantum electrodynamics, which would be deprived of the above limitations, and therefore it might be also interesting to consider the evolution of nodial lines in certain exact nonlinear theory as Born-Infeld electrodynamics. However, in this case, one cannot expect to find the polynomial solutions as given by Eqs. (17) and (21) and only numerical calculations come into play. This situation is similar to that in nonlinear quantum mechanics.

At the end we would like to note that although the observed deformation and evolution of vortices have their roots in the quantum nature of the vacuum, similar structures may also appear in classical and linear fields by the appropriate perturbation of the vortex configurations. Both the deviation of a vortex ring from the planar character as well as the occurrence of a 'hairpin'-shaped vortex, similar to that of Fig. 5, are known in optical diffraction [37].

[1] M. R. Matthews *et al.*, Phys. Rev. Lett. **83**, 2498 (1999).
[2] K. W. Madison *et al.*, Phys. Rev. Lett. **84**, 806 (2000).
[3] F. Chevy, K. W. Madison, and J. Dalibard, Phys. Rev. Lett. **85**, 2223 (2000).
[4] B. P. Anderson *et al.*, Phys. Rev. Lett. **85**, 2857 (2000).
[5] E. P. Gross, Nuovo Cimento **20**, 454 (1961).
[6] L. P. Pitaevskii, **40**, 646 (1961) [Sov. Phys. JETP **13**, 451 (1961)].
[7] A. L. Fetter and J. D. Walecka, *Quantum Theory of Many-Particle Systems* (McGraw-Hill, New York, 1971).
[8] O. Madelung, Z. Phys. **40**, 342 (1926).
[9] F. Lund, Phys. Lett. A **159**, 245 (1991).
[10] S. Rica and E. Tirapegui, Physica D **61**, 246 (1992).
[11] S. Rica and E. Tirapegui, Phys. Rev. Lett. **64**, 878 (1990).
[12] J. Koplik and H. Levine, Phys. Rev. Lett. **71**, 1375 (1993).
[13] J. Koplik and H. Levine, Phys. Rev. Lett. **76**, 4745 (1996).
[14] A. A. Svidzinsky and A. L. Fetter, Phys. Rev. A **62**, 063617 (2000).
[15] A. Aftalion and T. Riviere, Phys. Rev. A **64**, 043611 (2001).
[16] J. J. Garcia-Ripoll and V. M. Pérez-Garcia, Phys. Rev. A **64**, 053611 (2001).
[17] J. J. Garcia-Ripoll *et al.*, Phys. Rev. Lett. **87**, 140403 (2001).
[18] I. Białynicki-Birula and Z. Białynicka-Birula, Phys. Rev. A **65**, 014101 (2001).
[19] I. Białynicki-Birula, Z. Białynicka-Birula, and C. Śliwa, Phys. Rev. A **61**, 032110 (2000).
[20] M. Berry, Found. Phys. **31**, 659 (2001).
[21] M. V. Berry and M. R. Dennis, Proc. R. Soc. London, Ser. A **457**, 2251 (2001).
[22] M. V. Berry and M. R. Dennis, J. Phys. A **34**, 8877 (2001).
[23] I. Białynicki-Birula *et al.*, Acta Phys. Pol. A **100**, 29 (2001).
[24] J. F. Nye, Proc. R. Soc. London, Ser. A **387**, 105 (1983).
[25] J. F. Nye, Proc. R. Soc. London, Ser. A **389**, 279 (1983).
[26] M. R. Dennis, *Topological Singularities in Wave Fields*, Ph.D. thesis, Bristol, 2001.
[27] I. Białynicki-Birula and Z. Białynicka-Birula, Phys. Rev. A **67**, 062114 (2003).
[28] I. Białynicki-Birula, J. Opt. A (to be published).
[29] Iwo Białynicki-Birula, Prog. Opt. **36**, 245 (1996), and references therein.
[30] M. V. Berry and M. R. Dennis, J. Opt. A (to be published).
[31] H. Euler and W. Heisenberg, Z. Phys. **98**, 714 (1932).
[32] J. Schwinger, Phys. Rev. **82**, 664 (1951).
[33] M. Born and L. Infeld, Proc. R. Soc. London, Ser. A **147**, 522 (1934); **150**, 141 (1935).
[34] I. Białynicki-Birula and Z. Białynicka-Birula, *Quantum Electrodynamics* (Pergamon, Oxford, 1975).
[35] M. V. Berry, J. Opt. A (to be published).
[36] G. J. Kaiser, J. Opt. A (to be published).
[37] M. V. Berry, J. F. Nye, and F. J. Wright, Philos. Trans. R. Soc. London, Ser. A **291**, 453 (1979).
[38] This difference is truly of quantum origin and is not connected with the mistaken sign in the formula (19b) in Ref. [27], which is only a literal error (private communication).